1. **(A)** The scouts bought $1000/5 = 200$ groups of five candy bars at a total cost of $200 \cdot 2 = 400$ dollars. They sold $1000/2 = 500$ groups of two candy bars for a total of $500 \cdot 1 = 500$ dollars. Their profit was $\$500 - \$400 = \$100$.

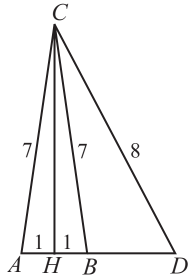
2. **(D)** We have
$$\frac{x}{100} \cdot x = 4, \quad \text{so} \quad x^2 = 400.$$
Because $x > 0$, it follows that $x = 20$.

3. **(C)** The number of CDs that Brianna will finally buy is three times the number she has already bought. The fraction of her money that will be required for all the purchases is $(3)(1/5) = 3/5$. The fraction she will have left is $1 - 3/5 = 2/5$.

4. **(B)** To earn an A on at least 80% of her quizzes, Lisa needs to receive an A on at least $(0.8)(50) = 40$ quizzes. Thus she must earn an A on at least $40 - 22 = 18$ of the remaining 20. So she can earn a grade lower than an A on at most 2 of the remaining quizzes.

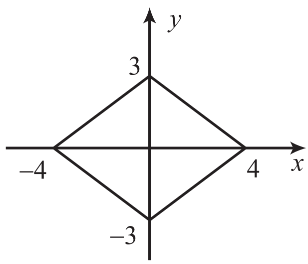
5. **(A)** The four white quarter circles in each tile have the same area as a whole circle of radius 1/2, that is, $\pi(1/2)^2 = \pi/4$ square feet. So the area of the shaded portion of each tile is $1 - \pi/4$ square feet. Since there are $8 \cdot 10 = 80$ tiles in the entire floor, the area of the total shaded region in square feet is
$$80\left(1 - \frac{\pi}{4}\right) = 80 - 20\pi.$$

6. **(A)** Let $\overline{CH}$ be an altitude of $\triangle ABC$. Applying the Pythagorean Theorem to $\triangle CHB$ and to $\triangle CHD$ produces
$$8^2 - (BD+1)^2 = CH^2 = 7^2 - 1^2 = 48, \quad \text{so} \quad (BD+1)^2 = 16.$$
Thus $BD = 3$.



7. **(D)** The graph is symmetric with respect to both coordinate axes, and in the first quadrant it coincides with the graph of the line $3x + 4y = 12$. Therefore the region is a rhombus, and the area is

$$\text{Area} = 4\left(\frac{1}{2}(4 \cdot 3)\right) = 24.$$



8. **(C)** The vertex of the parabola is $(0, a^2)$. The line passes through the vertex if and only if $a^2 = 0 + a$. There are two solutions to this equation, namely $a = 0$ and $a = 1$.

9. **(B)** The percentage of students getting 95 points is

$$100 - 10 - 25 - 20 - 15 = 30,$$

so the mean score on the exam is

$$0.10(70) + 0.25(80) + 0.20(85) + 0.15(90) + 0.30(95) = 86.$$

Since fewer than half of the scores were less than 85, and fewer than half of the scores were greater than 85, the median score is 85. The difference between the mean and the median score on this exam is $86 - 85 = 1$.

10. **(E)** The sequence begins $2005, 133, 55, 250, 133, \ldots$. Thus after the initial term 2005, the sequence repeats the cycle 133, 55, 250. Because $2005 = 1 + 3 \cdot 668$, the 2005^{th} term is the same as the last term of the repeating cycle, 250.

11. **(D)** There are

$$\binom{8}{2} = \frac{8!}{6! \cdot 2!} = 28$$

ways to choose the bills. A sum of at least \$20 is obtained by choosing both \$20 bills, one of the \$20 bills and one of the six smaller bills, or both \$10 bills. Hence the probability is

$$\frac{1 + 2 \cdot 6 + 1}{28} = \frac{14}{28} = \frac{1}{2}.$$

12. **(D)** Let r_1 and r_2 be the roots of $x^2+px+m=0$. Since the roots of $x^2+mx+n=0$ are $2r_1$ and $2r_2$, we have the following relationships:

$$m = r_1r_2, \quad n = 4r_1r_2, \quad p = -(r_1+r_2), \quad \text{and} \quad m = -2(r_1+r_2).$$

So

$$n = 4m, \quad p = \frac{1}{2}m, \quad \text{and} \quad \frac{n}{p} = \frac{4m}{\frac{1}{2}m} = 8.$$

OR

The roots of

$$\left(\frac{x}{2}\right)^2 + p\left(\frac{x}{2}\right) + m = 0$$

are twice those of $x^2+px+m=0$. Since the first equation is equivalent to $x^2+2px+4m=0$, we have

$$m = 2p \quad \text{and} \quad n = 4m, \quad \text{so} \quad \frac{n}{p} = 8.$$

13. **(D)** Since $4^{x_1}=5$, $5^{x_2}=6, \ldots, 127^{x_{124}}=128$, we have

$$4^{7/2} = 128 = 127^{x_{124}} = \left(126^{x_{123}}\right)^{x_{124}} = 126^{x_{123}\cdot x_{124}} = \cdots = 4^{x_1x_2\cdots x_{124}}.$$

So $x_1x_2\cdots x_{124} = 7/2$.

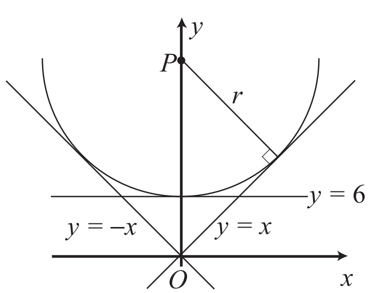
OR

We have

$$\begin{aligned} x_1x_2\cdots x_{124} &= \log_4 5\cdot\log_5 6\cdots\log_{127}128 \\ &= \frac{\log 5}{\log 4}\cdot\frac{\log 6}{\log 5}\cdots\frac{\log 128}{\log 127} = \frac{\log 128}{\log 4} = \frac{\log 2^7}{\log 2^2} = \frac{7\log 2}{2\log 2} = \frac{7}{2}. \end{aligned}$$

14. **(E)** Let O denote the origin, P the center of the circle, and r the radius. A radius from the center to the point of tangency with the line $y=x$ forms a right triangle with hypotenuse $\overline{OP}$. This right triangle is isosceles since the line $y=x$ forms a 45° angle with the y-axis. So

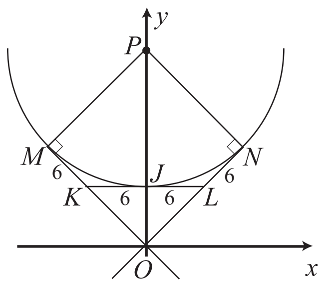
$$r\sqrt{2} = r+6 \quad \text{and} \quad r = \frac{6}{\sqrt{2}-1} = 6\sqrt{2}+6.$$



OR

Let the line $y = -x$ intersect the circle and the line $y = 6$ at M and K, respectively, and let the line $y = x$ intersect the circle and the line $y = 6$ at N and L, respectively. Quadrilateral $PMON$ has four right angles and $MP = PN$, so $PMON$ is a square. In addition, $MK = KJ = 6$ and $KO = 6\sqrt{2}$. Hence

$$r = MO = MK + KO = 6 + 6\sqrt{2}.$$



15. **(D)** The sum of the digits 1 through 9 is 45, so the sum of the eight digits is between 36 and 44, inclusive. The sum of the four units digits is between $1+2+3+4 = 10$ and $6+7+8+9 = 30$, inclusive, and also ends in 1. Therefore the sum of the units digits is either 11 or 21. If the sum of the units digits is 11, then the sum of the tens digits is 21, so the sum of all eight digits is 32, an impossibility. If the sum of the units digits is 21, then the sum of the tens digits is 20, so the sum of all eight digits is 41. Thus the missing digit is $45 - 41 = 4$. Note that the numbers 13, 25, 86, and 97 sum to 221.

OR

Each of the two-digit numbers leaves the same remainder when divided by 9 as does the sum of its digits. Therefore the sum of the four two-digit numbers leaves the same remainder when divided by 9 as the sum of all eight digits. Let d be the missing digit. Because 221 when divided by 9 leaves a remainder of 5, and the sum of the digits from 1 through 9 is 45, the number $(45 - d)$ must leave a remainder of 5 when divided by 9. Thus $d = 4$.

16. **(D)** The centers of the unit spheres are at the 8 points with coordinates $(\pm1, \pm1, \pm1)$, which are at a distance

$$\sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$$

from the origin. Hence the maximum distance from the origin to any point on the spheres is $1 + \sqrt{3}$.

17. **(B)** The given equation is equivalent to

$$\log_{10}\left(2^a \cdot 3^b \cdot 5^c \cdot 7^d\right) = 2005, \quad \text{so} \quad 2^a \cdot 3^b \cdot 5^c \cdot 7^d = 10^{2005} = 2^{2005} \cdot 5^{2005}.$$

Let M be the least common denominator of a, b, c and d. It follows that

$$2^{Ma} \cdot 3^{Mb} \cdot 5^{Mc} \cdot 7^{Md} = 2^{2005M} \cdot 5^{2005M}.$$

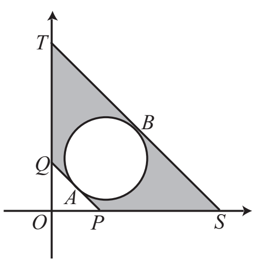
Since the exponents are all integers, the Fundamental Theorem of Arithmetic implies that

$$Ma = 2005M, \quad Mb = 0, \quad Mc = 2005M, \quad \text{and} \quad Md = 0.$$

Hence the only solution is $(a, b, c, d) = (2005, 0, 2005, 0)$.

18. **(C)** For $\triangle ABC$ to be acute, all angles must be acute. For $\angle A$ to be acute, point C must lie above the line passing through A and perpendicular to $\overline{AB}$. The segment of that line in the first quadrant lies between $P(4, 0)$ and $Q(0, 4)$. For $\angle B$ to be acute, point C must lie below the line through B and perpendicular to $\overline{AB}$. The segment of that line in the first quadrant lies between $S(14, 0)$ and $T(0, 14)$. For $\angle C$ to be acute, point C must lie outside the circle U that has $\overline{AB}$ as a diameter. Let O denote the origin. Region R, shaded below, has area equal to

$$\begin{aligned}\text{Area}(\triangle OST) - \text{Area}(\triangle OPQ) - \text{Area}(\text{Circle } U) &= \frac{1}{2} \cdot 14^2 - \frac{1}{2} \cdot 4^2 - \pi\left(\frac{\sqrt{50}}{2}\right)^2 \\ &= 90 - \frac{25}{2}\pi \approx 51.\end{aligned}$$



19. **(E)** By the given conditions, it follows that $x > y$. Let $x = 10a + b$ and $y = 10b + a$, where $a > b$. Then

$$m^2 = x^2 - y^2 = (10a + b)^2 - (10b + a)^2 = 99a^2 - 99b^2 = 99(a^2 - b^2).$$

Since $99(a^2 - b^2)$ must be a perfect square,

$$a^2 - b^2 = (a + b)(a - b) = 11k^2,$$

for some positive integer k. Because a and b are distinct digits, we have $a - b \leq 9 - 1 = 8$ and $a + b \leq 9 + 8 = 17$. It follows that $a + b = 11$, $a - b = k^2$, and k is either 1 or 2.

If $k = 2$, then $(a, b) = (15/2, 7/2)$, which is impossible. Thus $k = 1$ and $(a, b) = (6, 5)$. This gives $x = 65$, $y = 56$, $m = 33$, and $x + y + m = 154$.

20. **(C)** Note that the sum of the elements in the set is 8. Let $x = a+b+c+d$, so $e + f + g + h = 8 - x$. Then

$$(a+b+c+d)^2 + (e+f+g+h)^2 = x^2 + (8-x)^2$$
$$= 2x^2 - 16x + 64 = 2(x-4)^2 + 32 \geq 32.$$

The value of 32 can be attained if and only if $x = 4$. However, it may be assumed without loss of generality that $a = 13$, and no choice of b, c, and d gives a total of 4 for x. Thus $(x - 4)^2 \geq 1$, and

$$(a+b+c+d)^2 + (e+f+g+h)^2 = 2(x-4)^2 + 32 \geq 34.$$

A total of 34 can be attained by letting a, b, c, and d be distinct elements in the set $\{-7, -5, 2, 13\}$.

21. **(C)** Let $n = 7^k Q$, where Q is the product of primes, none of which is 7. Let d be the number of divisors of Q. Then n has $(k+1)d$ divisors. Also $7n = 7^{k+1}Q$, so $7n$ has $(k + 2)d$ divisors. Thus

$$\frac{(k+2)d}{(k+1)d} = \frac{80}{60} = \frac{4}{3} \quad \text{and} \quad 3(k+2) = 4(k+1).$$

Hence $k = 2$. Note that $n = 2^{19}7^2$ meets the conditions of the problem.

22. **(E)** Note that

$$z_{n+1} = \frac{iz_n}{\overline{z_n}} = \frac{iz_n^2}{z_n\overline{z_n}} = \frac{iz_n^2}{|z_n|^2}.$$

Since $|z_0| = 1$, the sequence satisfies

$$z_1 = iz_0^2, \; z_2 = iz_1^2 = i\left(iz_0^2\right)^2 = -iz_0^4,$$

and, in general, when $k \geq 2$,

$$z_k = -iz_0^{2^k}.$$

Hence z_0 satisfies the equation $1 = -iz_0^{(2^{2005})}$, so $z_0^{(2^{2005})} = i$. Because every nonzero complex number has n distinct nth roots, this equation has 2^{2005} solutions. So there are 2^{2005} possible values for z_0.

OR

Define

$$\operatorname{cis}\theta = \cos\theta + i\sin\theta.$$

Then if $z_n = r \operatorname{cis} \theta$ we have

$$z_{n+1} = \frac{\operatorname{cis}(\theta + 90^\circ)}{\operatorname{cis}(-\theta)} = \operatorname{cis}(2\theta + 90^\circ).$$

The first terms of the sequence are $z_0 = \operatorname{cis}\alpha$, $z_1 = \operatorname{cis}(2\alpha + 90^\circ) = iz_0^2$, $z_2 = \operatorname{cis}(4\alpha + 270^\circ) = \operatorname{cis}(4\alpha - 90^\circ) = \frac{z_0^4}{i}$, $z_3 = \operatorname{cis}(8\alpha - 90^\circ) = \frac{z_0^8}{i}$, and, in general,

$$z_n = \frac{z_0^{(2^n)}}{i} \quad \text{for } n \geq 2.$$

So

$$z_{2005} = \frac{z_0^{(2^{2005})}}{i} = 1 \quad \text{and} \quad z_0^{(2^{2005})} = i.$$

As before, there are 2^{2005} possible solutions for z_0.

23. **(B)** From the given conditions it follows that

$$x + y = 10^z, \quad x^2 + y^2 = 10 \cdot 10^z \quad \text{and} \quad 10^{2z} = (x+y)^2 = x^2 + 2xy + y^2.$$

Thus

$$xy = \frac{1}{2}(10^{2z} - 10 \cdot 10^z).$$

Also

$$(x+y)^3 = 10^{3z} \quad \text{and} \quad x^3 + y^3 = (x+y)^3 - 3xy(x+y),$$

which yields

$$\begin{aligned} x^3 + y^3 &= 10^{3z} - \frac{3}{2}(10^{2z} - 10 \cdot 10^z)(10^z) \\ &= 10^{3z} - \frac{3}{2}(10^{3z} - 10 \cdot 10^{2z}) = -\frac{1}{2}10^{3z} + 15 \cdot 10^{2z}, \end{aligned}$$

and $a + b = -\frac{1}{2} + 15 = 29/2$.

No other value of $a + b$ is possible for all members of S, because the triple $\left(\frac{1}{2}(1+\sqrt{19}), \frac{1}{2}(1-\sqrt{19}), 0\right)$ is in S, and for this ordered triple, the equation $x^3 + y^3 = a \cdot 10^{3z} + b \cdot 10^{2z}$ reduces to $a + b = 29/2$.

24. **(A)** Suppose that the triangle has vertices $A(a, a^2)$, $B(b, b^2)$ and $C(c, c^2)$. The slope of line segment $\overline{AB}$ is

$$\frac{b^2 - a^2}{b - a} = b + a,$$

so the slopes of the three sides of the triangle have a sum

$$(b + a) + (c + b) + (a + c) = 2 \cdot \frac{m}{n}.$$

The slope of one side is $2 = \tan\theta$, for some angle θ, and the two remaining sides have slopes

$$\tan\left(\theta \pm \frac{\pi}{3}\right) = \frac{\tan\theta \pm \tan(\pi/3)}{1 \mp \tan\theta\tan(\pi/3)} = \frac{2 \pm \sqrt{3}}{1 \mp 2\sqrt{3}} = -\frac{8 \pm 5\sqrt{3}}{11}.$$

Therefore

$$\frac{m}{n} = \frac{1}{2}\left(2 - \frac{8+5\sqrt{3}}{11} - \frac{8-5\sqrt{3}}{11}\right) = \frac{3}{11},$$

and $m + n = 14$.

Such a triangle exists. The x-coordinates of its vertices are $(11 \pm 5\sqrt{3})/11$ and $-19/11$.

OR

Define the vertices as in the first solution, with the added stipulations that $a < b$ and $\overline{AB}$ has slope 2. Then

$$2 = \frac{b^2 - a^2}{b - a} = b + a, \quad \text{so} \quad a = 1 - k \text{ and } b = 1 + k,$$

for some $k > 0$. If D is the midpoint of $\overline{AB}$, then

$$D = \left(1, \frac{(1-k)^2 + (1+k)^2}{2}\right) = (1, 1 + k^2).$$

The slope of the altitude $\overline{CD}$ is $-1/2$, so

$$1 - c = 2(c^2 - 1 - k^2).$$

Therefore

$$CD^2 = (1-c)^2 + \left(c^2 - 1 - k^2\right)^2 = \frac{5}{4}(1-c)^2.$$

Because $\triangle ABC$ is equilateral, we also have

$$CD^2 = \frac{3}{4}AB^2 = \frac{3}{4}\left((2k)^2 + (4k)^2\right) = 15k^2.$$

Hence

$$\frac{5}{4}(1-c)^2 = 15k^2, \quad \text{so} \quad k^2 = \frac{(1-c)^2}{12}.$$

Substitution into the equation $1 - c = 2(c^2 - 1 - k^2)$ yields $c = 1$ or $c = -19/11$. Because $c < 1$, it follows that

$$a + b + c = 2 - \frac{19}{11} = \frac{3}{11} = \frac{m}{n}, \quad \text{so} \quad m + n = 14.$$

25. **(A)** Because each ant can move from its vertex to any of four adjacent vertices, there are 4^6 possible combinations of moves. In the following, consider only those combinations in which no two ants arrive at the same vertex. Label the vertices as A, B, C, A', B' and C', where A', B' and C' are opposite A, B and C, respectively. Let f be the function that maps each ant's starting vertex onto its final vertex. Then neither of $f(A)$ nor $f(A')$ can be either A or A', and similar statements hold for the other pairs of opposite vertices. Thus there are $4 \cdot 3 = 12$ ordered pairs of values for $f(A)$ and $f(A')$. The vertices $f(A)$ and $f(A')$ are opposite each other in four cases and adjacent to each other in eight.

Suppose that $f(A)$ and $f(A')$ are opposite vertices, and, without loss of generality, that $f(A) = B$ and $f(A') = B'$. Then $f(C)$ must be either A or A' and $f(C')$ must be the other. Similarly, $f(B)$ must be either C or C' and $f(B')$ must be the other. Therefore there are $4 \cdot 2 \cdot 2 = 16$ combinations of moves in which $f(A)$ and $f(A')$ are opposite each other.

Suppose now that $f(A)$ and $f(A')$ are adjacent vertices, and, without loss of generality, that $f(A) = B$ and $f(A') = C$. Then one of $f(B)$ and $f(B')$ must be C' and the other cannot be B'. So there are four possible ordered pairs of values for $f(B)$ and $f(B')$. For each of those there are two possible ordered pairs of values for $f(C)$ and $f(C')$. Therefore there are $8 \cdot 4 \cdot 2 = 64$ combinations of moves in which $f(A)$ and $f(A')$ are adjacent to each other.

Hence the probability that no two ants arrive at the same vertex is

$$\frac{16+64}{4^6} = \frac{5 \cdot 2^4}{2^{12}} = \frac{5}{256}.$$